Kleene's Theorem states the equivalence of the following three statements:

1. A language is regular (i.e., is represented by a regular expression).
2. A language is accepted by a NDFA.
3. A language is accepted by a FA.

In the textbook by Cohen, he states the theorem using TG's in place of NDFAs. It makes no difference. We could add a fourth statement to the list, but Kleene did not. In trying to stay close to the text, I will restate Kleene's Theorem using TGs, and also as a set of implications.

Restatement of Kleene's Theorem:

1. If a language is regular , there is a TG that accepts it.
2. If a language is accepted by a TG, then there is a FA that accepts it.
3. If a language is accepted by a FA, then it is regular (i.e., there is a regular expression that defines it.

In these notes, I prove statements 2 and 3 above. The proof of statement 1 is very easy and will be added at a later date.

Proof of 2. For any TG M, there is a FA M' such that L(M') = L(M).

This is a constructive proof. Given a TG M, it defines a FA M' that accepts the same language as the TG.

Let M have states $s_1, s_2, ..., s_n$ and assume that the set of start states of M is S and the set of final states is F.

First, create a TG M_0 exactly like M except that M_0 has a unique start state. M_0 is identical to M except that it has a new start state, s_0, with Λ-transitions to each state of M that is in S, and the states that were start states in M are non-start states in M_0. Since any word accepted by M can be accepted by M_0 by using a Λ-transition to enter the same start state that would lead to its acceptance in M, and since any word that is accepted by must be accepted by M since it must first reach a state that is a start state in M, without M_0 reading any letters, $L(M_0) = L(M)$.

Next, let M_1 be exactly the same as M_0 except that it contains no edges labeled by strings of length greater than 1. To do this, first let M_1 be a copy of M_0. Then, for each edge in M_1 that is labeled by a string of length > 1, do the following. Suppose the edge from s_i to s_j in M_1 is labeled by $w = a_1a_2a_3...a_k$, where $k > 1$. Create k-1 new states in M_1 with unique labels, say $t_1, t_2, ..., t_{k-1}$, create the transitions $\delta(s_i,a_1) = t_1$, $\delta(t_1,a_2) = t_2$, $\delta(t_2,a_3) = t_3$, ..., $\delta(t_{k-2},a_{k-1}) = t_{k-1}$, and $\delta(t_{k-1},a_k) = s_j$, and delete the edge from s_i to s_j. Then M_1 accepts the same language as M_0 because M_0 can move from s_i to s_j on w if and only if M_1 can move from s_i to s_j on w by entering the new intermediate states.

M_1 is now a NDFA with Λ-transitions, called a NDFA-Λ. The next step is to build the FA M' that accepts the same language as M_1. For any state s, define

$$\Lambda-closure(s) \quad = \quad \{t \;\mid\; \delta(s,\Lambda)=t \quad \vee \quad (\exists u)(u \in \Lambda-closure(s) \quad \wedge \quad \delta(u,\Lambda)=t)\}$$

Notice that this is a recursive definition of the Λ-closure. The recursion is embedded within the curly braces, but it is nonetheless recursion. In plain words, the Λ-closure of a state s is the set of states that a NDFA-Λ can enter from s without reading any symbols. Now define the Λ-closure of a set of states S:

$$\Lambda-closure(S) \quad = \quad \bigcup_{s \in S} \Lambda-closure(s)$$

We can now construct the FA M'. The idea is that the states of M' will be sets of states from M_1. The following pseudo-code algorithm constructs the FA M'.

Let s_0 be the unique start state of M_1.
Let S_0 = Λ-closure(s_0) be the start state of M'.
Let Q denote the collection of states of M'. Add S_0 to Q and mark it *unprocessed*.
while there is a state-set S in Q that is *unprocessed* do
    mark S *processed*;
    for each input symbol a do
        Let T be the set of all states to which there is a transition on 'a' from some state in S;
        Let T_λ = Λ-closure(T);
        if T_λ is not in Q then
            add T_λ to Q and mark it *unprocessed*;
        add a transition from S to T_λ labeled 'a';
For each state-set S in Q, if S contains a final state of M, make S a final state of M'.

Claim: L(M') = L(M_1).

Proof.

Let δ(s,a) denote the transition function of M_1. Since is a NDFA, δ(s,a) is the *set* of states that can be entered by M_1 on reading 'a' in state s. The emphasis is on "set" because the transition function is not the same as that of a FA -- it defines a set. *The set δ(s,a) includes any states that it can reach by following the Λ-transitions in M_1.* By definition, δ*(s,w) is the set of states that can be reached by M_1 on reading the string w in state s, again including the possibility that it might have used Λ-transitions.

Let $\delta_{M'}$(S,a) denote the transition function of M'. From the algorithm above, the transition function $\delta_{M'}$(S,a) is defined by

$$\delta_{M'}(S,a) \;=\; \bigcup_{s \in S} \delta(s,a) \tag{1}$$

because the definition includes the states entered by all Λ transitions. That is why the Λ-closure is computed at each step.

Claim: For any string w,

$$\delta^*_{M'}(S, w) = \bigcup_{s \in S} \delta^*(s, w) \tag{2}$$

This can be proved by induction on the length of w. It is true for |w| = 0 since

$$\delta^*_{M'}(S, \Lambda) = S = \bigcup_{s \in S} \delta^*(s, \Lambda) \tag{3}$$

because the states in M' are their own Λ-closures, so it follows from the definition of Λ-closure. Assume it is true for any w with |w| = m and let w be a word of length m+1. Then w = va, where |v| = m. Hence

$$\begin{aligned}
\delta^*_{M'}(S, w) &= \delta^*_{M'}(S, va) \\
&= \delta_{M'}\left(\delta^*_{M'}(S, v), a\right) \\
&= \delta_{M'}\left(\bigcup_{s \in S} \delta^*(s, v), a\right) \\
&= \bigcup_{s \in S} \delta\left(\delta^*(s, v), a\right) \\
&= \bigcup_{s \in S} \delta^*(s, va) \\
&= \bigcup_{s \in S} \delta^*(s, w)
\end{aligned}$$

The second step used the definition of $\delta_{M'}^*$ and the third step applied the inductive hypothesis on v. The fourth step used the definition from (1) (and an implicit step I have not included, but which can be proved easily enough.) The last two steps follow from the definition of w and δ^*. It follows that the claim is proved.

Since S_0 = Λ-closure(s_0) is the start state of M',

$$\delta^*_{M'}(S_0, w) = \delta^*(s_0, w)$$

Also, since w is in L(M') if and only if $\delta_{M'}^*$ (S_0,w) is a final state, from the above, w is in L(M') if and only if $\delta^*(s_0,w)$ contains a final state in M_1, which is true if and only if w is in L(M_1).

Proof of 3. If a language L is accepted by some FA, then there is a regular expression r such that L = <r>.

Let L be accepted by an FA M with states s_1, s_2, ..., s_n. Assume that s_1 is the start state of M and that the set of final states of M is denoted F. Define the set L(i,j,k) to be the set of all words that cause M, starting in state s_j to enter state s_j without passing through any of the states s_{k+1}, s_{k+2}, ..., s_n. In other words, L(i,j,k) is the set of words that start in s_i and end in s_j and *pass through* only the states s_1, s_2, ..., s_k. "Passing through" means entering and leaving, like one does in a toll booth or turnstile. It does not mean "landing there" and staying there.

The language accepted by M is the set of all words that cause M, when starting in state s_1 to stop in a final state, passing through any of the states of M. This means that, if s_f is a final state, then L(1,f,n) consists only of words accepted by M, and that

$$L = L(1,f_1, n) \cup L(1,f_2, n) \cup \ldots \cup L(1,f_m, n) \quad (4)$$

where $F = \{s_{f1}, s_{f2}, \ldots, s_{fm}\}$.

From the definition of L(i,j,k) it follows that, for each i and j, 1 <= i, j <= n, L(i,j,0) is the set of all symbols that label the transitions from s_i to s_j, and that in addition, if i = j, then the null string is also in this set. Formally,

$$L(i,j,0) = \begin{cases} \{a \mid \delta(\sigma_i, a) = \sigma_j\} \cup \Lambda & \text{if } i = j \\ \{a \mid \delta(\sigma_i, a) = \sigma_j\} & \text{if } i \neq j \end{cases} \quad (5)$$

Furthermore, for all k > 0, the set L(i,j,k) can be defined recursively from the following observation. (I will use the language abusively and talk about a word starting in a state or passing through a state or even visiting a state. What this means of course is that the word causes M to enter a state while reading it, or causes M to pass through a state while reading it, and so on.)

1. If a word starts in state s_i and terminates in state s_j without going through any states s_{k+1}, s_{k+2}, ..., s_n , then it falls into one of two cases:

2. It starts in state s_i and terminates in state s_j without going through any states s_k, s_{k+1}, ..., s_n , or

It starts in state s_i and terminates in state s_j and enters state s_k, and then visits other states without passing through any of s_{k+1}, s_{k+2}, ..., s_n , possibly passing through sk many times, and then returns to s_k for the last time, and then travels a path to state s_j.

In short, either the word was already in L(i,j,k-1) (Case 1) or it is in L(i,j,k) but not in L(i,j,k-1), and is there because it passes through state s_k, and we can break the word into 3 pieces: the "left"piece x that first reaches s_k without going through any states s_k, s_{k+1}, ..., s_n, the "middle" piece y that travels around M without going through any states s_k, s_{k+1}, ..., s_n until it visits s_k for the last time, and the "right" piece z that reaches s_j from s_k without going through any states s_k, s_{k+1}, ..., s_n. Since x in in L(i,k,k-1), y is in L(k,k,k-1)* and y is in L(k,j,k-1), it follows that

$$L(i,j,k) = L(i,j,k-1) \cup L(i,k,k-1) \cdot L(k,k,k-1)^* \cdot L(k,j,k-1) \quad (6)$$

Claim: For every i and j, 1 <= i,j <= n, and for every k, 0 <= k <= n, the set L(i,j,k) can be represented by a regular expression.

Proof.

We can prove this by induction on k.

For each i and j, the set L(i,j,0) is a finite set and is therefore regular. Let r(i,j, 0) denote the regular expression such that L(i,j,0) = < r(i,j, 0) >.

Assume that the claim is true for k-1. Then, for any i and j, there exists a regular expression that we can denote r(i,j,k-1) such that L(i,j,k-1) = <r(i,j,k-1)>. From formula (6) and the induction hypothesis it follows that

$$\begin{aligned} L(i,j,k) &= \langle r(i,j,k-1)\rangle + \langle r(i,k,k-1)\rangle\cdot\langle r(k,k,k-1)^*\rangle\cdot\langle r(k,j,k-1)\rangle \\ &= \langle r(i,j,k-1) + r(i,k,k-1)\cdot r(k,k,k-1)^*\cdot r(k,j,k-1)\rangle \end{aligned} \qquad (7)$$

where each of r(i,j,k-1), r(i,k,k-1), r(k,k,k-1), and r(k, j, k-1) is a regular expression. Since the right hand side is a regular expression, it follows that L(i,j,k) is a regular language , and that we can let r(i,j,k) denote the regular expression that defines it. By the axiom of induction, it is true for all k >= 0. Of course, for k > n, the sets do not change since there are no states in the FA numbered higher than s_n, so although in principle all of these sets exist, we are only concerned about the ones for which k <= n. **QED**.

The truth of the theorem follows from formulas (4) and (7). Formula (4) states that L is a finite union of the sets L(1,s,n) for which s is a final state of M, and formula (7) states that each of the sets L(1,s,n) can be represented by regular expressions, so that

$$L = \langle r(1,f_1, n) + r(1,f_2, n) + \ldots + r(1,f_m, n) \rangle \qquad (8)$$

proving that L is a regular expression. **QED**.

The proof of the theorem implicitly defines a tabular algorithm that can be used to construct the regular expression. It also suggests a recursive function that can be used to construct the expression. The most efficient solution, however, would be a dynamic programming solution, combining the simple and inefficient table-driven approach with the recursive solution. I will not describe that algorithm here. For now, I present a recursive algorithm , written in C with pseudo-code.

Let M have states 1, 2, 3, ..., n. Assume the alphabet is Σ. Assume that δ(i,a) is the transition function, which can also be represented by a 2D matrix δ[i,a].

The main function is *BuildRE()*, which takes the FA, and integers i, j, and k, and constructs a string re that contains the regular expression, fully parenthesized to avoid possible ambiguities. The FA is used inside the function in pseudo-code that looks up all symbols that cause a transition from state i to state j. I leave out necessary declarations and such.

```
void BuildRE ( FA M, int i, int j, int k, char re[] )
{
    char re1[MAXSIZE];
    char re2[MAXSIZE];
    char re3[MAXSIZE];
    char re4[MAXSIZE];

    if ( k == 0 ) {
        re = { a in SIGMA | M.delta(i,a) == j };
        if ( i == j )
            re = re + 'LAMBDA';
    }
    else { // k > 0
        BuildRE ( M, i, j, k-1, re1);
        BuildRE ( M, i, k, k-1, re2);
        BuildRE ( M, k, k, k-1, re3);
        BuildRE ( M, k, j, k-1, re4);
        sprintf (re, "(%s)+(%s)(%s)*(%s)", re1, re2, re3, re4 );
    }
}
```

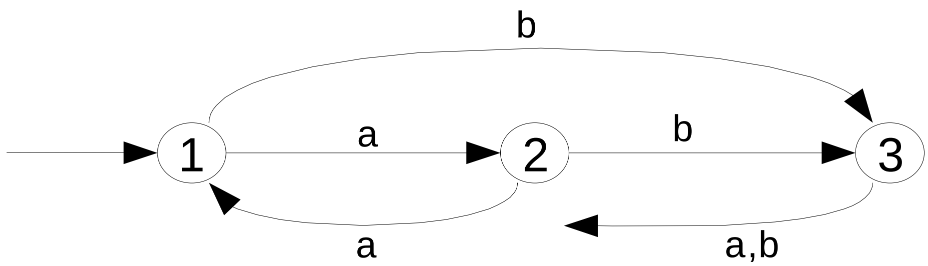
The main program is simply

```
void main ()
{

    sprintf(re, "()");
    for ( i = 1; i <= n; i++ )
        if ( finalstate(i) ) {
            BuildRE( M, 1, i, n, temp_re);
            sprintf(re, "(%s)+(%s)", re, temp_re);
        }
    printf("%s\n", re);
}
```

Example

We will build the regular expression for the FA below using a table-driven method.



	k			
i,j	0	1	2	3
1,1	Λ	Λ	(aa)*	
1,2	a	a	a(aa)*	a(aa)* + a*b ((a+b) a*b)*(a+b)(aa)*
1,3	b	b	a*b	a*b ((a+b) a*b)*
2,1	a	a	a(aa)*	
2,2	Λ	Λ + aa	(aa)*	
2,3	b	b + ab	a*b	
3,1	∅	∅	(a + b)(aa)*a	
3,2	a + b	a + b	(a + b)(aa)*	
3,3	Λ	Λ	Λ + (a + b)a*b	

Note. L(1,3,3) is simplified from a*b + a*b(Λ + (a+b)a*b)* (Λ + (a+b)a*b). There is no need to calculate any other parts of the table. Since L(M) is the union of L(1,2,3) and L(1,3,3), the final expression is

L(M) = a(aa)* + a*b ((a+b) a*b)* (Λ + (a+b)(aa)*)