1. Prove, using the Pumping Lemma, that \( \{a^nb^{2n} | n > 0 \} \) is not regular.

Let \( N \) be the constant of the lemma. Let \( w \) be the word \( a^N b^{2N} \). By the lemma, there exist \( x, y, \) and \( z \) such that \( w = xyz \), \( |xy| <= N \), \( |y| > 0 \), and for all \( k \), \( x^k y^k z \) is in \( L \). Since \( |xy| <= N \), \( y \) consists entirely of \( a \)'s. Let \( |y| = m \). By the lemma, the string \( xz \) is in \( L \), and \( xz = a^{N-m} b^{2N} \). But \( 2N != 2(N-m) \), since \( m > 0 \), so this is a contradiction. Therefore, this language cannot be regular.

2. Let \( L = \{ wa^n | w \in \{a,b,c\}^* \} \). In other words, \( L \) consists of words \( wa^n \) where \( w \) contains \( a \)'s, \( b \)'s, and \( c \)'s and \( n \) is the length of \( w \).

i. Use the Myhill-Nerode Theorem to prove that \( L \) is not regular.

Consider the sequence of strings \( b, bb, bbb, ... \) for all \( k > 0 \). Pick any two of them, say \( b^k \) and \( b^m \), where \( k != m \). Then the word \( b^k a^k \) is in \( L \) but \( b^m a^k \) is not in \( L \). Therefore, no two of the words in this infinite sequence are in the same equivalence class, proving that \( L \) must have an infinite number of such classes. Therefore, \( L \) is not regular, by the Myhill-Nerode theorem.

ii. Use the Pumping Lemma to prove \( L \) is not regular.

Let \( N = 6 \). For any word \( w \) in \( L \) whose length is at least 6, we can write \( w = xyz \), where \( x \) is the null string, \( y = aa \), and \( z \) is the rest of \( w \). Note that \( |xy| <= 6 \) and \( |y| = 2 > 0 \). Because \( w \) is in \( L \) and its length is not a prime number, its length is an even number. Since \( |y| = 2 \), \( |xz| \) is an even number and cannot be 2, and for any \( k \),
|xy^kz| = |w| + 2k must also be an even number, implying it is not a prime number and hence \(xy^kz\) is in \(L\).

4. Give an example of a regular language \(R\) and a non-regular language \(L\) such that \(R + L\) is regular, and prove or justify that \(R + L\) is regular.

This is easy – let \(R\) be \((a+b)^*\) and let \(L\) be any of the non-regular languages above. The union of \(R\) and \(L\) is \(R\), since \(R\) contains all languages over \(\{a,b\}\).

5. Give an example of a regular language \(R\) and a non-regular language \(L\) such that \(R + L\) is non-regular, and prove or justify that \(R + L\) is non-regular.

Let \(R\) be any finite language and let \(L\) be a language containing \(R\) that is not regular. Then \(R + L = L\) and \(L\) is not regular. As an example, let \(R = \{ a^2, a^3 \}\) and let \(L = PRIME\). \(PRIME\) contains \(R\).

6. Let \(L\) be a regular language over \(\Sigma = \{a,b\}\). Define \(L' = \{ x \mid \text{there exists} \ y \in \Sigma^* \text{ such that} \ xy \in L \}\). Is \(L'\) regular? Either prove it is or give an example to show it may not always be.

\(L'\) is regular. To see this, let \(M\) be a FA accepting \(L\). Let it have states \(Q = \{ q_1, q_2, ..., q_n \}\). Let \(F\) be the set of final states of \(M\). Let \(M'\) be a FA identical to \(M\) except for which states are final states. For each state \(q\) in \(Q\) for which there exists at least one word \(z\) such that \(\delta^*(q, z)\) is a final state in \(M\), make \(q\) a final state in \(M'\).

Let \(w\) be in \(L(M')\). Then \(w\) reaches a final state of \(M'\), which means that \(w\) is a word that reaches a state \(q\) in \(M\) such that there is a word \(y\) such that \(\delta^*(q, y)\) is in \(L\). This implies that \(wy \in \Sigma^*\). Therefore, \(w\) is in \(L'\).

Conversely, let \(w\) be in \(L'\). Then there is a \(y \in \Sigma^*\) such that \(wy\) is in \(L\). Let \(q\) be the state in \(M\) that \(w\) reaches. Then \(\delta^*(q, y)\) is a final state in \(M\), which means that \(q\) is a final state in \(M'\), and hence \(w\) is in \(L(M')\). This proves that \(M'\) accepts \(L'\).